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THE CONSTRUCTION OF PERIODIC ORBITS CLOSE TO TRIANGULAR LIBRATION POINTS FOR THE THREE-BODY PROBLEM IN A RESISTIVE MEDIUM†

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A circular, restricted three-body problem is considered when a passively gravitating point also experiences the action of small resistive forces, acting in the opposite direction to the absolute velocity vector. The nature of the loss of stability of the triangular libration points is studied using the Poincaré method in which the ratio of the magnitude of the resistive force to the force of gravitational attraction serves as the small parameter. Asymptotically stable periodic orbits are constructed. Two Lyapunov families of periodic orbits, which exist in the neighbourhood of the libration points of the classical theory, are the generating families. Calculations were carried out for mass ratios corresponding to the Earth–Jupiter and Earth–Moon systems, with different values of the parameters characterizing the law of resistance. © 2001 Elsevier Science Ltd. All rights reserved.

It has been shown [1, 2] that the positions of relative equilibrium, close to Lagrangian points, are conserved in the problem under consideration. Depending on the nature of the resistance (which is determined by the hypothetical distribution of interplanetary matter), these equilibria can be asymptotically stable or unstable. According to the general theory of dynamical systems, loss of stability following a change in the parameters of a system is accompanied by a bifurcation of the creation of the cycle. The *a priori* possibility of the existence of asymptotically stable periodic orbits in the neighbourhood of unstable libration points follows from this.

1. FORMULATION OF THE PROBLEM

We shall investigate the restricted three-body problem in a rarefied interplanetary medium which exerts a certain (small) resistance on moving objects. It has been noted [3] that, during the early stages of the solar system, the density of this medium was significantly higher than at present and its resistance could have played a noticeable role in the establishment of the present-day orbits: in particular, the secular decrease in the eccentricities of the Kepler ellipses is a consequence of this factor. In studying the effect of resistance on the triangular libration points [1, 2] it has been assumed that the main attracting bodies move in circular orbits around the common centre of mass and that, apart from gravitational forces, a resistance of the form

$$\mathbf{S} = -\varepsilon g(|\mathbf{V}|) f(r_1) \mathbf{V} \tag{1.1}$$

acts on the small mass, where $\mathbf{V}$ is the velocity in the absolute system of coordinates, r_1 is the distance to the Sun, f is a scalar function describing the density distribution of the matter, g is a function which specifies the dependence of the drag force on the velocity and ε is a small positive constant.

The equations of motion in a system of coordinates which rotates together with the main attracting bodies (angular velocity $\boldsymbol{\omega} = (0, 0, 1)$) have the following vector form [1]

$$\begin{gathered} \ddot{\mathbf{r}} = \operatorname{grad} U(\mathbf{r}) + \omega^2 \mathbf{r} - (\boldsymbol{\omega}, \mathbf{r})\boldsymbol{\omega} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \mathbf{S} \\ U(\mathbf{r}) = \frac{1-\mu}{r_1} + \frac{\mu}{r_2}, \quad \mathbf{r}_1 = \mathbf{r} + (\mu, 0, 0), \quad \mathbf{r}_2 = \mathbf{r} + (\mu - 1, 0, 0), \quad \mu = \frac{m_2}{m_1 + m_2} \end{gathered} \tag{1.2}$$

where m_1 and m_2 are the masses of the main attracting bodies.

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When $\varepsilon = 0$, system (1.2) possesses the equilibrium positions

$$L_4: \quad \mathbf{r}^\circ = \left(\tfrac{1}{2} - \mu, \ \frac{\sqrt{3}}{2}, \ 0\right), \quad L_5: \quad \mathbf{r}^\circ = \left(\tfrac{1}{2} - \mu, \ -\frac{\sqrt{3}}{2}, \ 0\right) \tag{1.3}$$

for which $r_1 = r_2 \equiv 1$. If the inequality

$$\mu < \mu^* = \tfrac{1}{2} - \sqrt{69}/18 = 0.0385\ldots \tag{1.4}$$

is satisfied, these equilibria are stable in the first approximation. Here, according to Lyapunov's theorem on a holomorphic integral, in the neighbourhood of each points (1.3), there are three (in accordance with the number of degrees of freedom of system (1.2)) families of periodic orbits. The first two of these correspond to the roots $\omega_1 > \omega_2 > 0$ of the defining equation

$$\omega^4 - \omega^2 + (27/4)M = 0, \quad M = \mu(1-\mu) \tag{1.5}$$

In this case, Lyapunov's theorems are not satisfied in the case of the second family for those values of μ for which

$$27M = \frac{4n^2}{(n^2+1)^2}, \quad n = 1, 2, \ldots \tag{1.6}$$

The orbits of the third family are orthogonal to the plane of motion of the main attracting bodies (see [4]).

For small values of $\varepsilon > 0$, the libration points are displaced by an amount of the order of magnitude of ε from the values (1.3) [1, 2]. In this case, the nature of their stability changes, since the roots of the characteristic equation converge with the imaginary axis. A consequence of this is that the Lyapunov families are also destroyed: since the perturbations are non-conservative, only those isolated orbits, for which the circulation of the drag forces is equal to zero, can be preserved. The construction of such orbits forms the subject of this paper.

2. PROCEDURE

The essence of Poincaré's method lies in the successive approximation of the periodic solution of a system with a small parameter ε with an error $O(\varepsilon^n)$. Here, as the initial approximation, one chooses the solution of the unperturbed system (that is, $\varepsilon = 0$) which satisfies the so-called amplitude equation [5]. This equation has a fairly simple meaning in natural mechanical systems which are subject to the action of small additional forces: the circulation of the forces for the periodic solution Γ of the unperturbed system must be equal to zero [5, 6]:

$$\oint_\Gamma (\mathbf{S}, d\mathbf{r}) = 0 \tag{2.1}$$

The restricted three-body problem under discussion possesses different types of periodic solutions (see [7] and the literature references therein). If the functions f and g in formula (1.1) are specified, the verification of condition (2.1) reduces to quadratures.

We shall only investigate the Lyapunov families close to the triangular libration points (1.3). In this case, the periodic orbits of the perturbed system can be considered in the plan of the bifurcation if the creation of the cycle from the equilibrium position with fixed ε and a change in the parameter μ or the parameters describing the type of resistance. Methods for investigating such a bifurcation have been described previously [8].

Two basic types of bifurcation in the creation of a cycle exist: sub-critical and super-critical. The first of these is characterized by the coexistence of a stable equilibrium and an unstable cycle; here loss of stability of the equilibrium is accompanied by the disappearance of the cycle. In the case of the second type a stable cycle is created with a loss of the stability of the equilibrium. In order to determine the type of bifurcation, it is sufficient to elucidate whether a root of the amplitude equation corresponds to the region of stability of the equilibrium position. The solution of this problem involves several stages.

1. For an arbitrary point on the boundary of the region of stability of a libration point, determine which of the roots of the characteristics equation lies on the imaginary axis. This enables one to determine which of the three Lyapunov families generates the limiting cycle.
2. It is then necessary to construct the required family of Lyapunov orbits in the neighbourhood of the libration points in a vacuum. Here, numerical calculations can be combined with analytical results [4].
3. On specifying the values of the parameters close to the boundary of the region of stability, calculations of the circulation of the drag forces are carried out along the periodic orbits of the classical problem. The results obtained enable one to determine the coefficients of the normal form of the equations of motion which are responsible for the nature of the bifurcation and to determine the roots of the amplitude equation. The generating orbits (the zeroth approximation in Poincaré's method) will thereby be constructed.

The generating orbits can be subsequently refined using Poincaré's method. This is not considered here.

3. RESULTS

We will now implement the scheme of investigation which has been described in the preceding section. We shall assume that inequality (1.4), which expresses the condition for the stability of the libration points when there is no resistance, is satisfied. In this case, the characteristic equation has three pairs of roots on the imaginary axis

$$\lambda_{1,2}^2 = -\tfrac{1}{2}(1+\sqrt{D}), \quad \lambda_{3,4}^2 = -\tfrac{1}{2}(1-\sqrt{D}), \quad \lambda_{5,6}^2 = -1, \quad D = 1-27M \tag{3.1}$$

When there is resistance of the form of (1.1), the eigenvalues of (3.1) are shifted off the imaginary axis by an amount of the order of ε; in this case, the third pair always transfers to the left half-plane while the position of the two remaining pairs depends on the form of the functions f and g.

The following condition

$$\sqrt{D}(1+\gamma) > |\beta(2-\mu) - \gamma\varkappa| \tag{3.2}$$

$$\gamma = \frac{V^\circ g'(V^\circ)}{2g(V^\circ)}, \quad \beta = \frac{f'(1)}{f(1)}, \quad V^\circ = \sqrt{1-M}, \quad \varkappa = 3 - \frac{9}{2}\frac{M}{1-M}$$

was obtained in [2] and, when this condition is satisfied, both these pairs fall in the left half-plane (in the more accurate form).

Inequality (3.2) has the identical form for both points L_4 and L_5; here, V° denotes the absolute velocity of the libration point. The real parts of the roots of the characteristic equation have the form

$$\operatorname{Re}\lambda_{1,3} = -\varepsilon f(1) g(V^\circ)((1+\gamma) \pm [\gamma\varkappa - \beta(2-\mu)]/\sqrt{D}) \tag{3.3}$$

In view of the presence, on the right-hand side of inequality (3.2), of the sign of the absolute magnitude, two cases are possible when it is destroyed. In the first of these cases

$$\sqrt{D}(1+\gamma) < \beta(2-\mu) - \gamma\varkappa \tag{3.4}$$

Here, by virtue of (3.3), we obtain $\operatorname{Re}\lambda_1 > 0$, $\operatorname{Re}\lambda_3 < 0$. Consequently, the creation of limiting cycles is only possible from short-period orbits (that is, from orbits from the Lyapunov family corresponding to the root λ_1).

In the second case

$$\sqrt{D}(1+\gamma) > \gamma\varkappa - \beta(2-\mu) \tag{3.5}$$

Here, we have, on the other hand, $\operatorname{Re}\lambda_1 < 0$, $\operatorname{Re}\lambda_3 > 0$, and the limiting cycle can therefore only be generated from the long-period family.

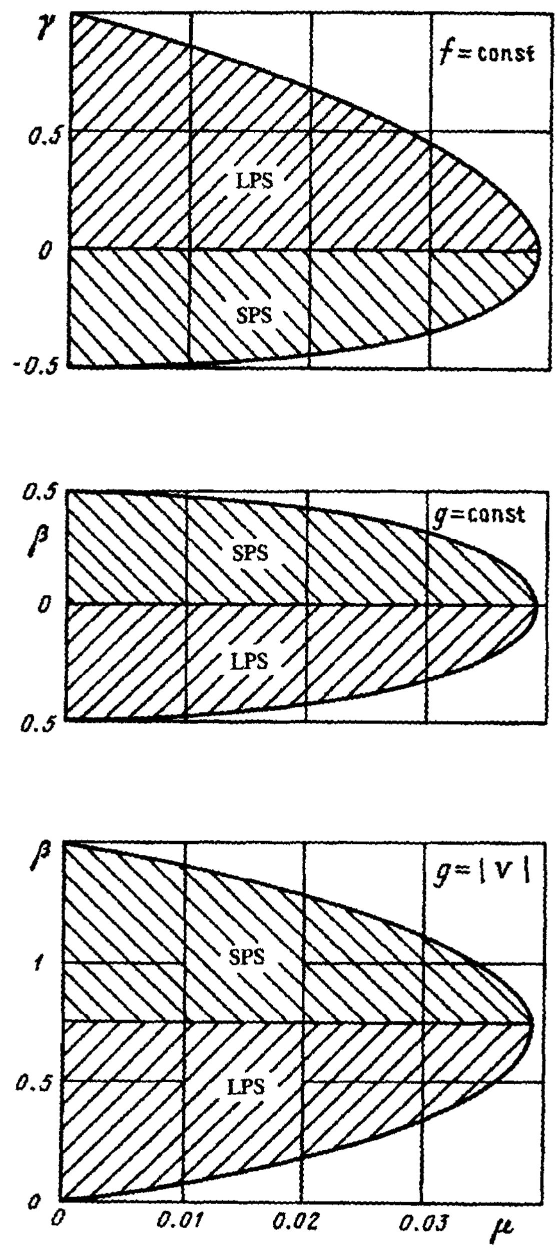

Fig. 1

According to the results obtained above, one should construct two families of Lyapunov orbits in the neighbourhoods of the triangular libration points, lying in the plane of motion of the main attracting bodies. The solution of this problem has been described in detail in [4]. The bifurcation in the creation of the cycle is described by the formula

$$\nu = Kl^2 + o(l^2) \tag{3.6}$$

where ν is a certain parameter which characterizes the removal of the representative point from the boundary of the region of stability and l is the length of the orbit.

The regions of stability of the triangular libration points in a medium with resistances of different kinds are shown the figure. Here, the hatching indicates which of the Lyapunov families undergoes bifurcation on reaching the boundary of the region. In the case of a medium of constant density (f = const. in Eq (1.1)), the parameter γ is determined by the formula

$$\gamma = V^\circ g'(V^\circ)/(2g(V^\circ))$$

were V° is the absolute velocity of the libration point. The bifurcation of the long-period solutions (LPS) is of a sub-critical nature while the bifurcation of the short-period solutions (SPS) is of a super-critical nature, that is, the stable limiting cycle is created from the SPS. The size of this cycle is determined using formula (3.6), in which $\nu = \gamma^* - \gamma$, $K = 6.0 \times 10^{-3}$ for the mass ratio $\mu = 9.5 \times 10^{-4}$ and $(S - J)$ and $K = 2.7 \times 10^{-3}$ for $\mu = 1.215 \times 10^{-2}$ (E − M). Here, S is the Sun, J is Jupiter, E is the Earth, M is the Moon and $\gamma^* = \gamma^*(\mu)$ is the value corresponding to the lower boundary of the region.

In cases where there is viscous friction (g = const in Eq. (1.1)) and aerodynamic resistance ($g = |\mathbf{V}|$), the coefficient β is given by the formula

$$\beta = |\mathbf{r}^\circ - \mathbf{r}_1|\, f'(|\mathbf{r}^\circ - \mathbf{r}_1|)/f(|\mathbf{r}^\circ - \mathbf{r}_1|)$$

where $\mathbf{r}^\circ$ is the radius vector of the libration point. The form of the bifurcations is analogous to the preceding form, $\nu = \beta - \beta^*$ and, for stable cycles, $K = 6.5 \times 10^{-3}$ (S − J) and $K = 3.3 \times 10^{-3}$ (E − M) in the case when g = const and $K = 7.2 \times 10^{-3}$ (S − J) and $K = 4.8 \times 10^{-3}$ (E − M) in the case when $g = |V|$. Here, $\beta^* = \beta^*(\mu)$ is the value corresponding to the upper boundary of the region.

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